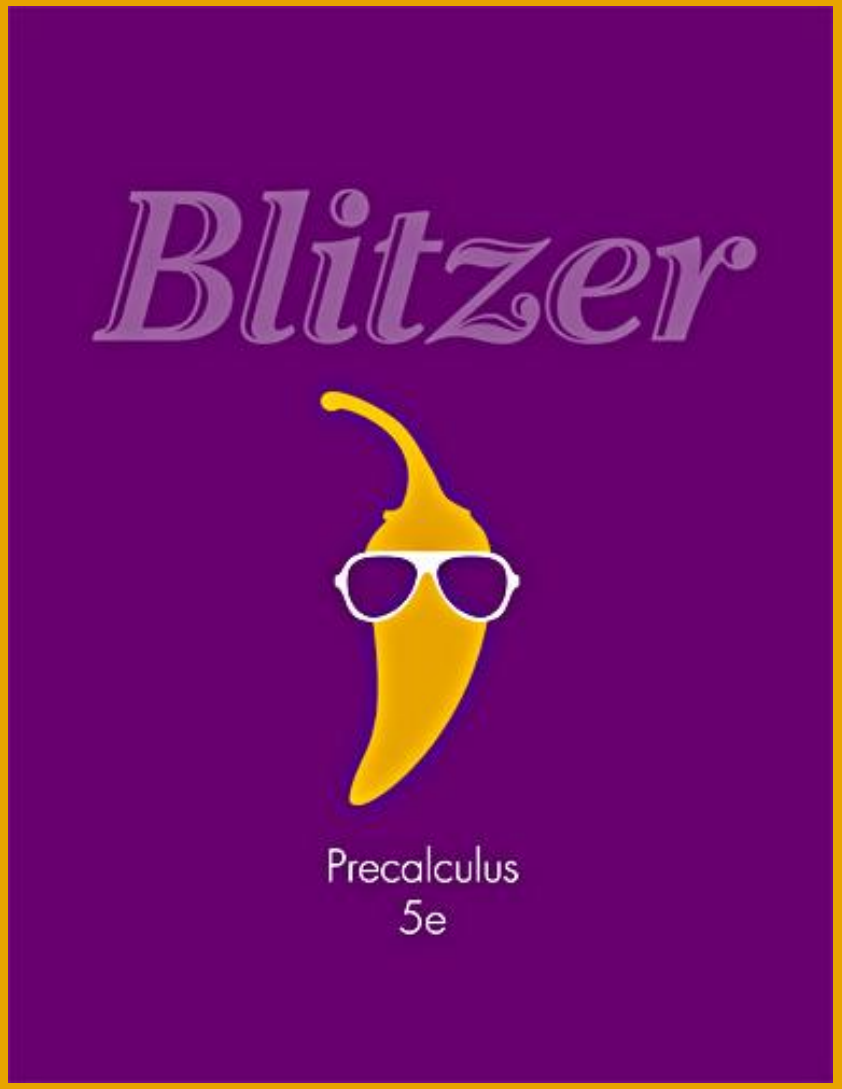
Chapter 4

Trigonometric Functions



4.1 Angles and Radian Measure

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Objectives:

- Recognize and use the vocabulary of angles.
- Use degree measure.
- Use radian measure.
- Convert between degrees and radians.
- Draw angles in standard position.
- Find coterminal angles.
- Find the length of a circular arc.
- Use linear and angular speed to describe motion on a circular path.

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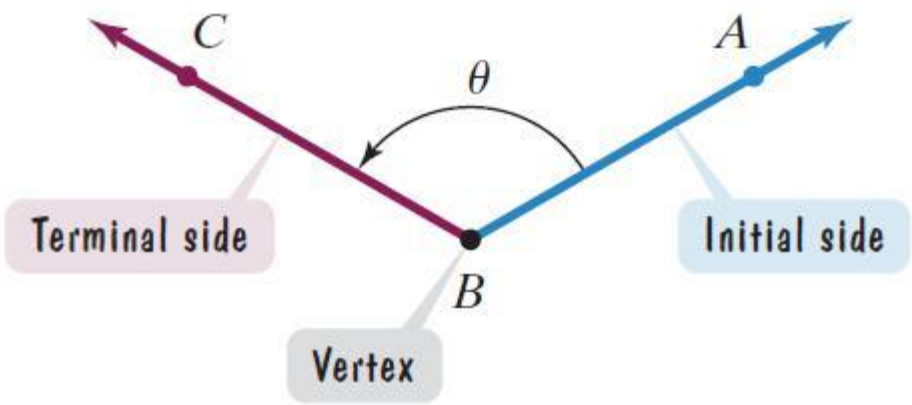
Angles

An **angle** is formed by two rays that have a common endpoint. One ray is called the **initial side** and the other the **terminal side**.



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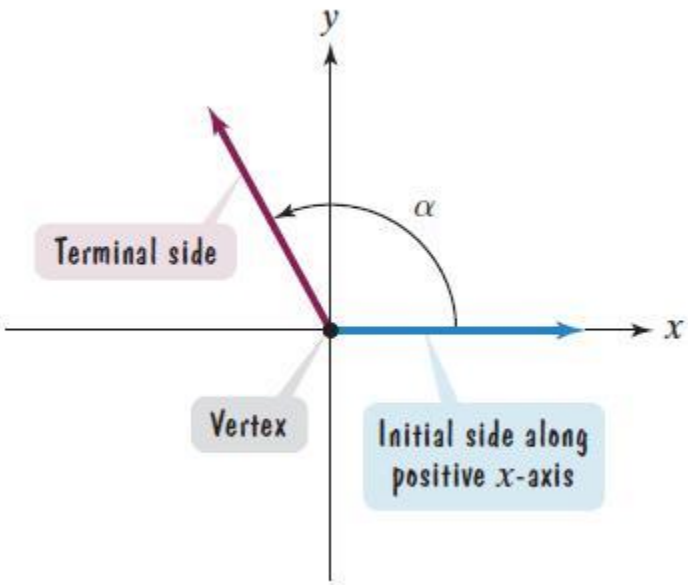
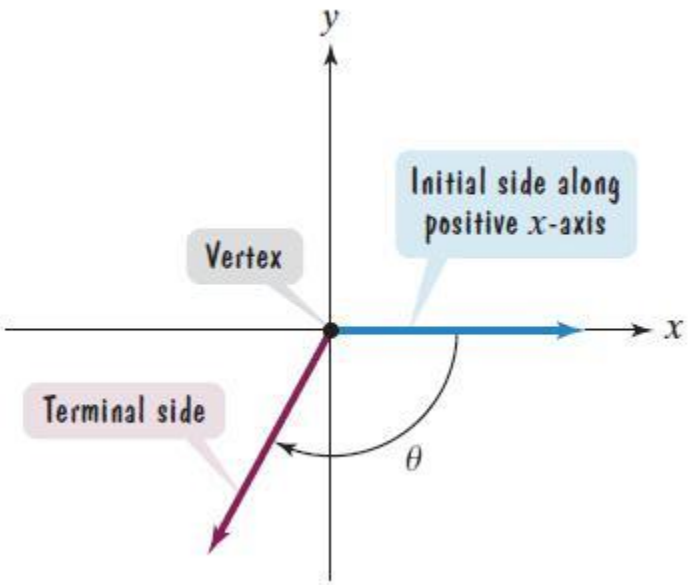
Angles *(continued)*

An angle is in standard position if its vertex is at the origin of a rectangular coordinate system and its initial side lies along the positive *x*-axis.





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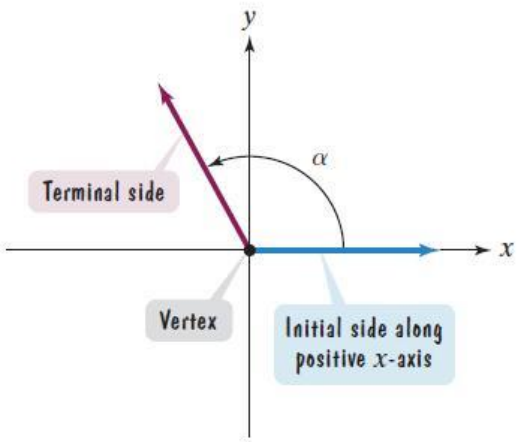
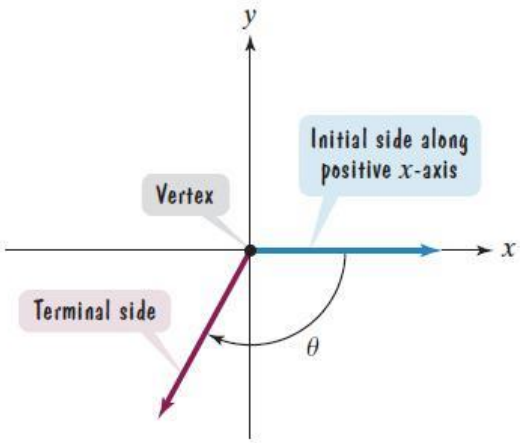
Angles *(continued)*

When we see an initial side and a terminal side in place, there are two kinds of rotations that could have generated the angle.

Positive angles are generated by counterclockwise rotation. Thus, angle α is positive. **Negative angles** are generated by clockwise rotation. Thus, angle θ is negative.





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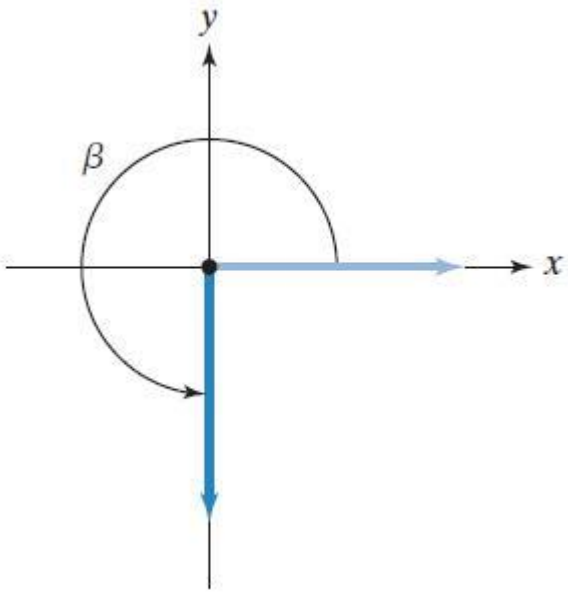
Angles *(continued)*

An angle is called a **quadrantal angle** if its terminal side lies on the *x*-axis or on the *y*-axis. Angle β is an example of a quadrantal angle.



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Measuring Angles Using Degrees

Angles are measured by determining the amount of rotation from the initial side to the terminal side. A complete rotation of the circle is 360 degrees, or 360°.

An **acute angle** measures less than 90°.

A **right angle** measures 90°.

An **obtuse angle** measures more than 90° but less than 180°.

A **straight angle** measures 180°.

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Measuring Angles Using Radians

An angle whose vertex is at the center of the circle is called a central angle. The **radian measure** of any central angle of a circle is the length of the intercepted arc divided by the circle's radius.

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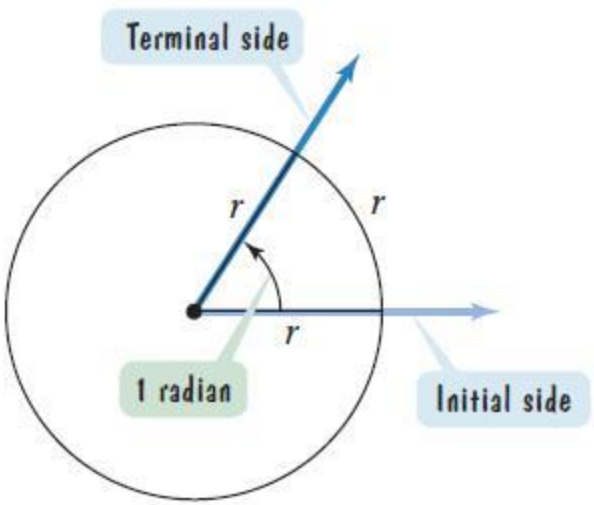
Definition of a Radian

One **radian** is the measure of the central angle of a circle that intercepts an arc equal in length to the radius of the circle.



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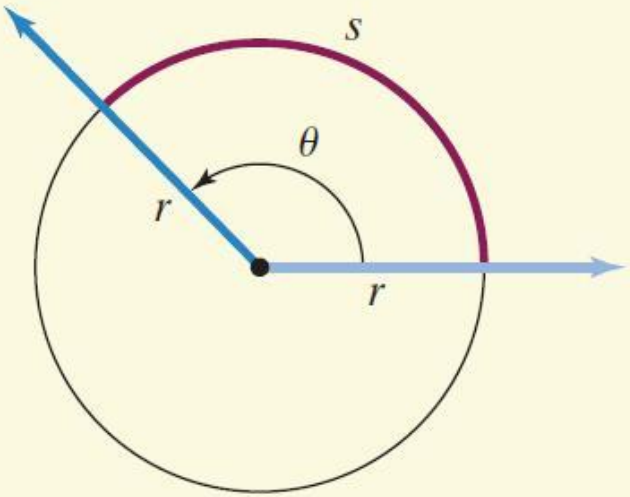
Radian Measure

Radian Measure

Consider an arc of length s on a circle of radius r. The measure of the central angle, θ, that intercepts the arc is

$$\theta = \frac{s}{r} \text{ radians.}$$



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Example: Computing Radian Measure

A central angle, θ, in a circle of radius 12 feet intercepts an arc of length 42 feet. What is the radian measure of θ?

$$\theta = \frac{s}{r} = \frac{42\ \cancel{\text{feet}}}{12\ \cancel{\text{feet}}} = 3.5$$

The radian measure of θ is 3.5 radians.

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Conversion between Degrees and Radians

Using the basic relationship π radians $= 180°$,

1. To convert degrees to radians, multiply degrees by $\dfrac{\pi \text{ radians}}{180°}$.

2. To convert radians to degrees, multiply radians by $\dfrac{180°}{\pi \text{ radians}}$.

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Example: Converting from Degrees to Radians

Convert each angle in degrees to radians:

a. $60° = 60\cancel{°}\,\square\,\dfrac{\pi \text{ radians}}{180\cancel{°}} = \dfrac{60\pi}{180}\text{radians} = \dfrac{\pi}{3}\text{radians}$

b. $270° = 270\cancel{°}\,\square\,\dfrac{\pi \text{ radians}}{180\cancel{°}} = \dfrac{270\pi}{180}\text{radians} = \dfrac{3\pi}{2}\text{radians}$

c. $-300° = -300\cancel{°}\,\square\,\dfrac{\pi \text{ radians}}{180\cancel{°}} = -\dfrac{300\pi}{180}\text{radians} = -\dfrac{5\pi}{3}\text{ radians}$

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Example: Converting from Radians to Degrees

Convert each angle in radians to degrees:

a. $\frac{\pi}{4}\text{ radians} = \frac{\cancel{\pi}\ \cancel{\text{radians}}}{4} \cdot \frac{180^\circ}{\cancel{\pi}\ \cancel{\text{radians}}} = \frac{180^\circ}{4} = 45^\circ$

b. $-\frac{4\pi}{3}\text{ radians} = -\frac{4\cancel{\pi}\ \cancel{\text{radians}}}{3} \cdot \frac{180^\circ}{\cancel{\pi}\ \cancel{\text{radians}}} = -\frac{4 \cdot 180^\circ}{3}$

$= -240^\circ$

c. $6\text{ radians} = 6\ \cancel{\text{radians}} \cdot \frac{180^\circ}{\pi\ \cancel{\text{radians}}} = \frac{6 \cdot 180^\circ}{\pi} \approx 343.8^\circ$

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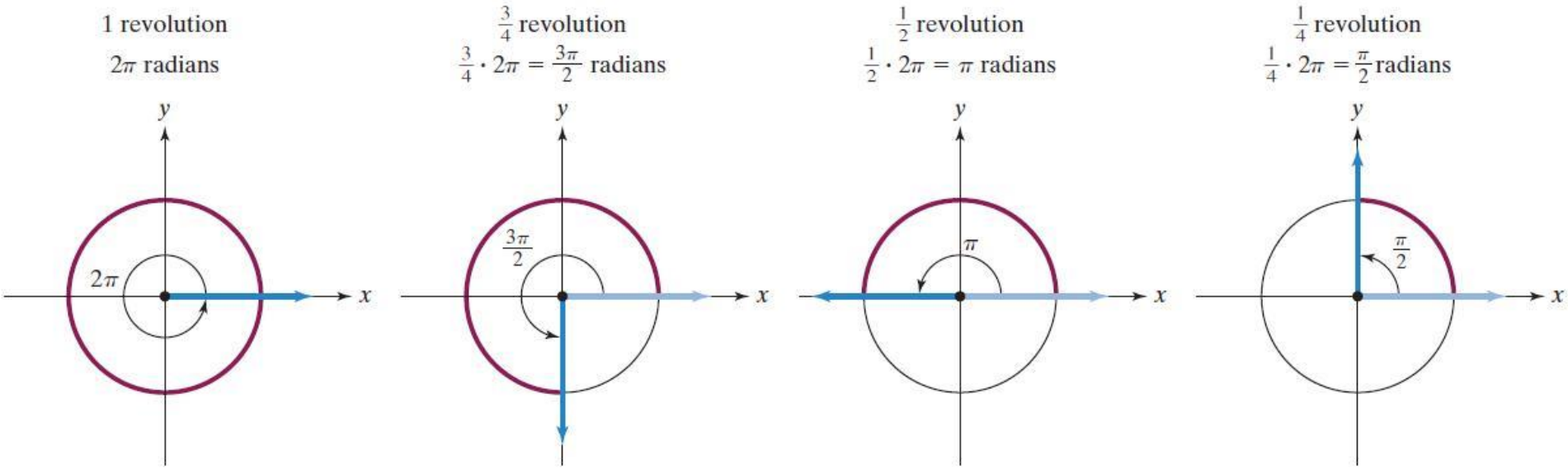
Drawing Angles in Standard Position

The figure illustrates that when the terminal side makes one full revolution, it forms an angle whose radian measure is 2π. The figure shows the quadrantal angles formed by 3/4, 1/2, and 1/4 of a revolution.



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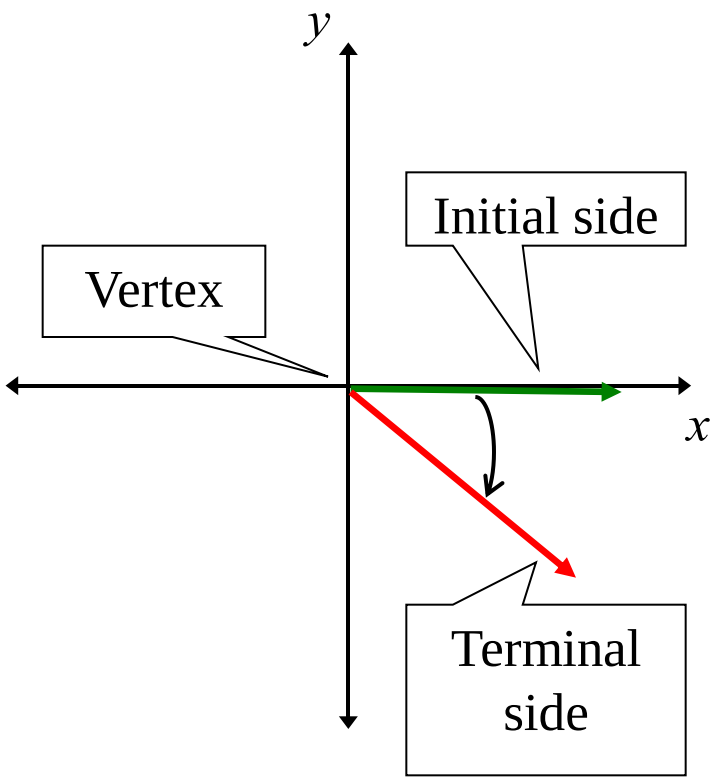
Example: Drawing Angles in Standard Position

Draw and label the angle in standard position: $\theta = -\frac{\pi}{4}$



The angle is negative. It is obtained by rotating the terminal side clockwise.

$$\left|-\frac{\pi}{4}\right| = \frac{1}{8}\cdot 2\pi$$

We rotate the terminal side clockwise $\frac{1}{8}$ of a revolution.

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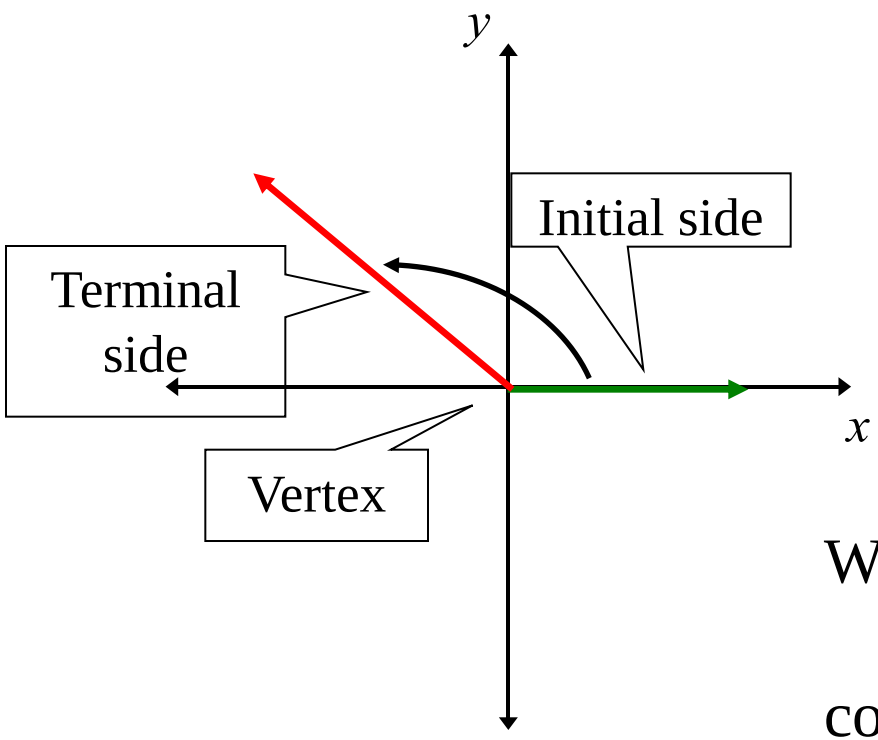
Example: Drawing Angles in Standard Position

Draw and label the angle in standard position: $\alpha = \dfrac{3\pi}{4}$



The angle is positive. It is obtained by rotating the terminal side counterclockwise.

$$\frac{3\pi}{4} = \frac{3}{8} \cdot 2\pi$$

We rotate the terminal side

counter clockwise $\dfrac{3}{8}$ of a revolution.

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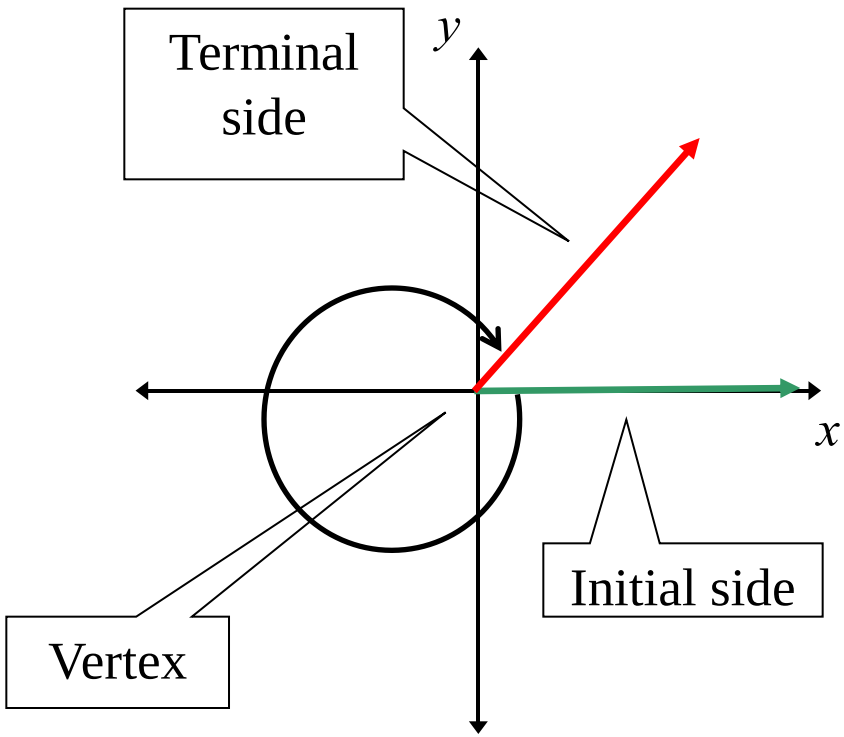
Example: Drawing Angles in Standard Position

Draw and label the angle in standard position: $\beta = -\frac{7\pi}{4}$



The angle is negative. It is obtained by rotating the terminal side clockwise.

$$\left|-\frac{7\pi}{4}\right| = \frac{7}{8}\cdot 2\pi$$

We rotate the terminal side clockwise $\frac{7}{8}$ of a revolution.

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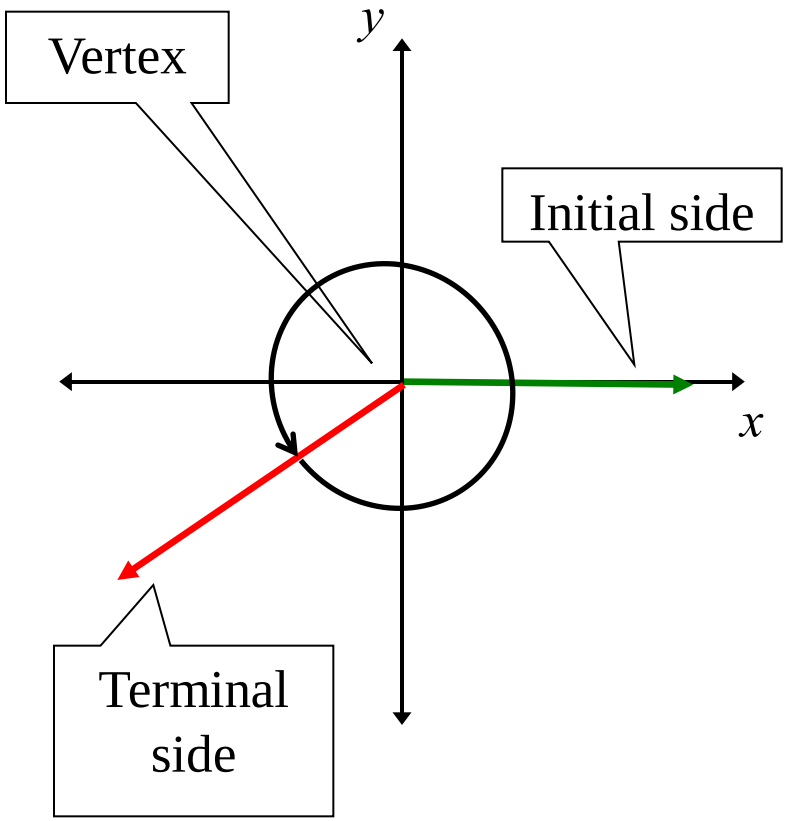
Example: Drawing Angles in Standard Position

Draw and label the angle in standard position: $\gamma = \dfrac{13\pi}{4}$



The angle is positive. It is obtained by rotating the terminal side counterclockwise.

$$\frac{13\pi}{4} = \frac{13}{8}\square 2\pi$$

We rotate the terminal side

counter clockwise $\dfrac{13}{8}$ of a revolution.

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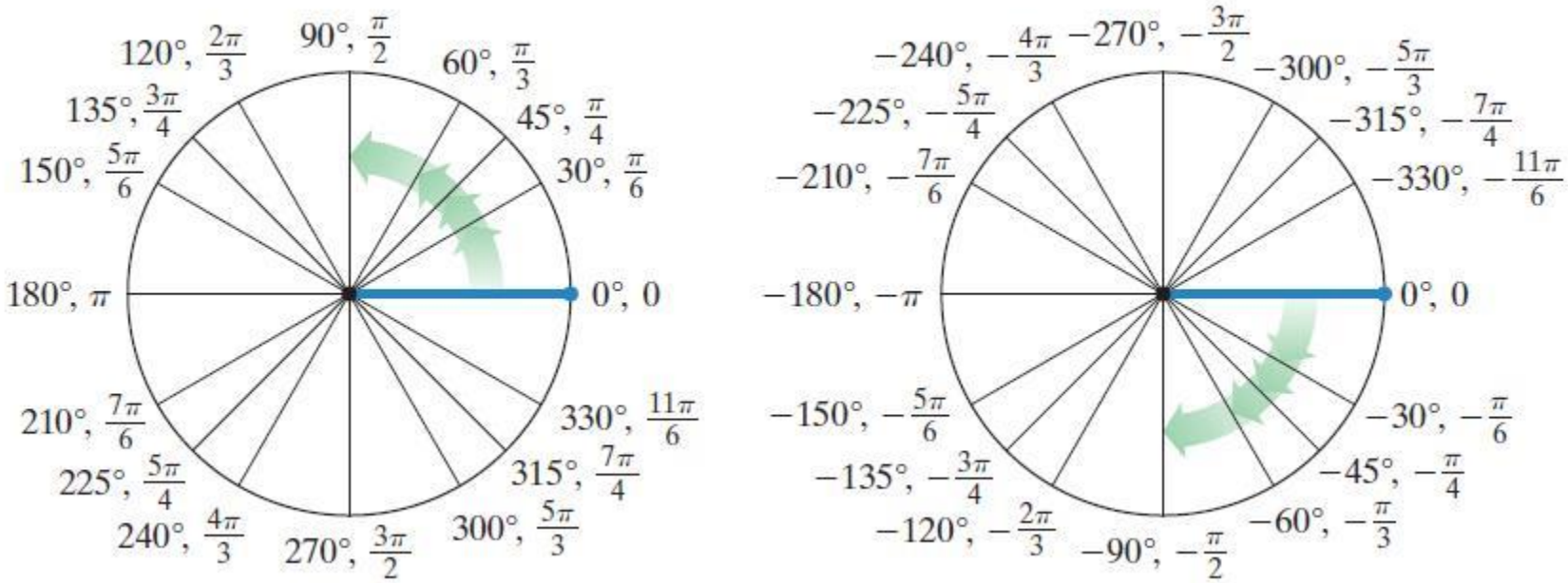
Degree and Radian Measures of Angles Commonly Seen in Trigonometry

In the figure below, each angle is in standard position, so that the initial side lies along the positive *x*-axis.



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Positive Angles in Terms of Revolutions of the Angle's Terminal Side Around the Origin

Terminal Side	Radian Measure of Angle	Degree Measure of Angle
$\frac{1}{12}$ revolution	$\frac{1}{12} \cdot 2\pi = \frac{\pi}{6}$	$\frac{1}{12} \cdot 360° = 30°$
$\frac{1}{8}$ revolution	$\frac{1}{8} \cdot 2\pi = \frac{\pi}{4}$	$\frac{1}{8} \cdot 360° = 45°$
$\frac{1}{6}$ revolution	$\frac{1}{6} \cdot 2\pi = \frac{\pi}{3}$	$\frac{1}{6} \cdot 360° = 60°$
$\frac{1}{4}$ revolution	$\frac{1}{4} \cdot 2\pi = \frac{\pi}{2}$	$\frac{1}{4} \cdot 360° = 90°$
$\frac{1}{3}$ revolution	$\frac{1}{3} \cdot 2\pi = \frac{2\pi}{3}$	$\frac{1}{3} \cdot 360° = 120°$

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Positive Angles in Terms of Revolutions of the Angle's Terminal Side Around the Origin *(continued)*

Terminal Side	Radian Measure of Angle	Degree Measure of Angle
$\frac{1}{2}$ revolution	$\frac{1}{2} \cdot 2\pi = \pi$	$\frac{1}{2} \cdot 360° = 180°$
$\frac{2}{3}$ revolution	$\frac{2}{3} \cdot 2\pi = \frac{4\pi}{3}$	$\frac{2}{3} \cdot 360° = 240°$
$\frac{3}{4}$ revolution	$\frac{3}{4} \cdot 2\pi = \frac{3\pi}{2}$	$\frac{3}{4} \cdot 360° = 270°$
$\frac{7}{8}$ revolution	$\frac{7}{8} \cdot 2\pi = \frac{7\pi}{4}$	$\frac{7}{8} \cdot 360° = 315°$
1 revolution	$1 \cdot 2\pi = 2\pi$	$1 \cdot 360° = 360°$

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Coterminal Angles

Two angles with the same initial and terminal sides but possibly different rotations are called **coterminal angles**.

Increasing or decreasing the degree measure of an angle in standard position by an integer multiple of 360° results in a coterminal angle. Thus, an angle of $\theta°$ is coterminal with angles of $\theta° \pm 360°k$, where k is an integer.

Increasing or decreasing the radian measure of an angle by an integer multiple of 2π results in a coterminal angle. Thus, an angle of θ radians is coterminal with angles of $\theta \pm 2\pi k$, where k is an integer.

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Example: Finding Coterminal Angles

Assume the following angles are in standard position. Find a positive angle less than 360° that is coterminal with each of the following:

a. a 400° angle

$$400° - 360° = 40°$$

b. a –135° angle

$$-135° + 360° = 225°$$

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Example: Finding Coterminal Angles

Assume the following angles are in standard position. Find a positive angle less than 2π that is coterminal with each of the following:

a. a $\frac{13\pi}{5}$ angle

$$\frac{13\pi}{5} - 2\pi = \frac{13\pi}{5} - \frac{10\pi}{5} = \frac{3\pi}{5}$$

b. a $-\frac{\pi}{15}$ angle

$$-\frac{\pi}{15} + 2\pi = -\frac{\pi}{15} + \frac{30\pi}{15} = \frac{29\pi}{15}$$

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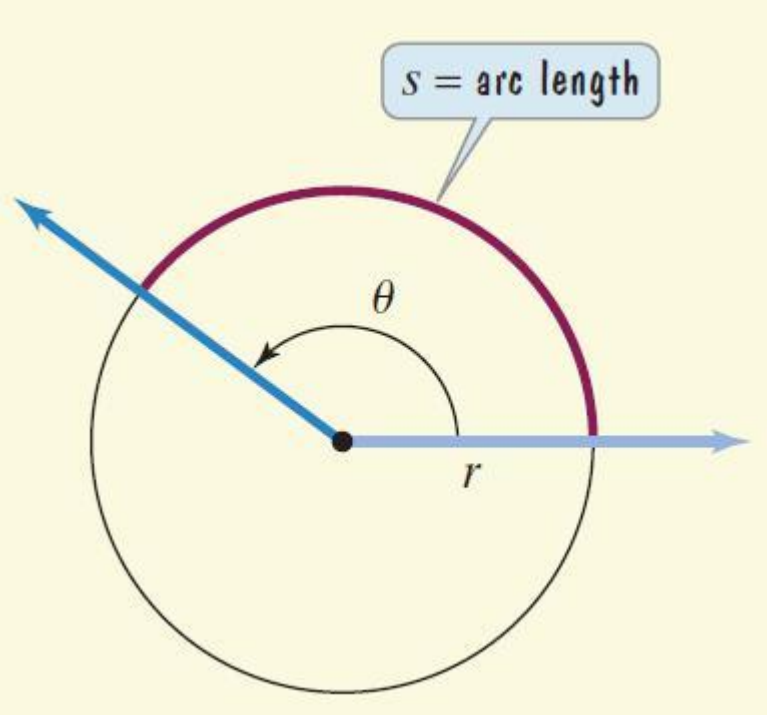
The Length of a Circular Arc

Let r be the radius of a circle and θ the nonnegative radian measure of a central angle of the circle. The length of the arc intercepted by the central angle is

$$s = r\theta.$$



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Example: Finding the Length of a Circular Arc

A circle has a radius of 6 inches. Find the length of the arc intercepted by a central angle of 45°. Express arc length in terms of π. Then round your answer to two decimal places.

We first convert 45° to radians:

$$45^\circ = 45\cancel{^\circ}\,\frac{\pi \text{ radians}}{180\cancel{^\circ}} = \frac{45\pi}{180} = \frac{\pi}{4} \text{ radians}$$

$$s = r\theta \quad = (6 \text{ inches})\left(\frac{\pi}{4}\right) = \frac{6\pi}{4} \text{ inches} \approx 4.71 \text{ inches.}$$

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Definitions of Linear and Angular Speed

If a point is in motion on a circle of radius r through an angle of θ radians in time t, then its **linear speed** is

$$v = \frac{s}{t},$$

where s is the arc length given by $s = r\theta$, and its **angular speed** is

$$\omega = \frac{\theta}{t}.$$

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Linear Speed in Terms of Angular Speed

The linear speed, v, of a point a distance r from the center of rotation is given by

$$v = r\omega,$$

where ω is the angular speed in radians per unit of time.

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Example: Finding Linear Speed

Long before iPods that hold thousands of songs and play them with superb audio quality, individual songs were delivered on 75-rpm and 45-rpm circular records. A 45-rpm record has an angular speed of 45 revolutions per minute. Find the linear speed, in inches per minute, at the point where the needle is 1.5 inches from the record's center.

Before applying the formula $\upsilon = r\omega$ we must express ω in terms of radians per second:

$$\omega = \frac{45 \text{ revolutions}}{1 \text{ minute}} \cdot \frac{2\pi \text{ radians}}{1 \text{ revolution}} = \frac{90\pi \text{ radians}}{1 \text{ minute}}$$

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Example: Finding Linear Speed *(continued)*

A 45-rpm record has an angular speed of 45 revolutions per minute. Find the linear speed, in inches per minute, at the point where the needle is 1.5 inches from the record's center.

The angular speed of the record is 90π radians per minute. The linear speed is

$$\upsilon = r\omega = 1.5 \text{ inches} \cdot \frac{90\pi}{1 \text{ minute}} = \frac{135\pi \text{ in}}{\text{min}} \approx \frac{424 \text{ in}}{\text{min}}$$

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