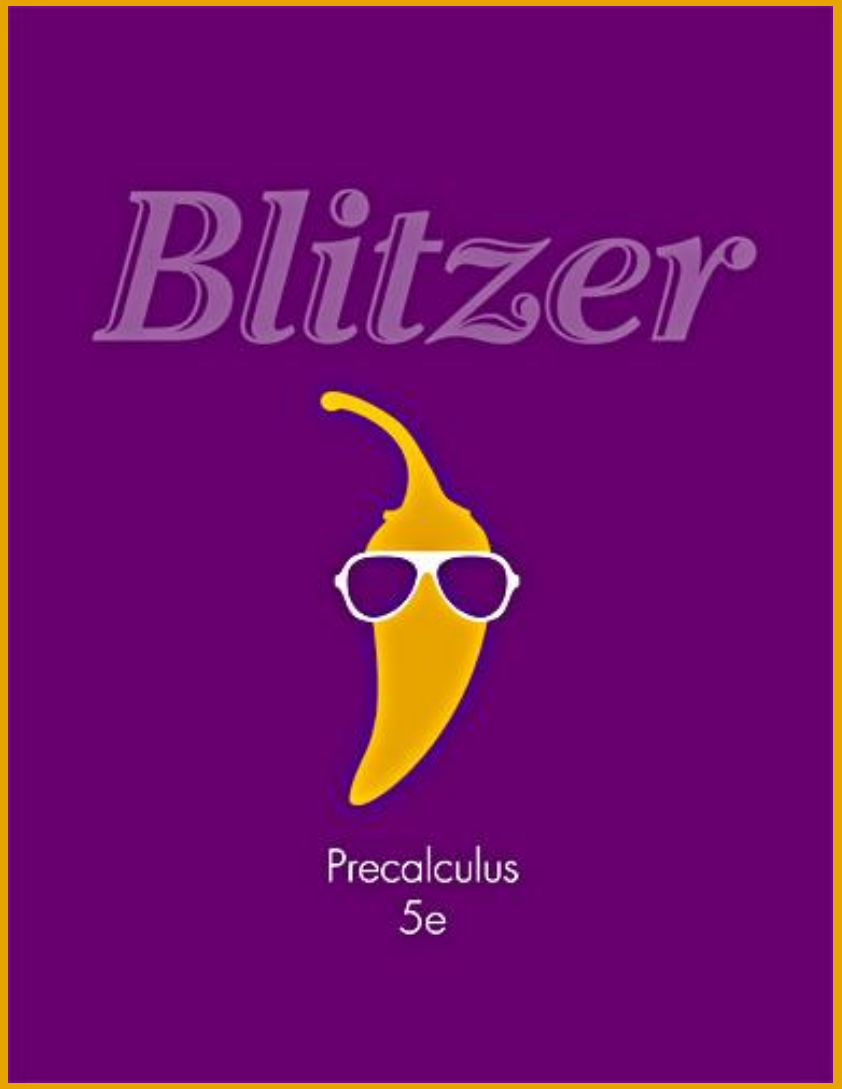
Chapter 6

Additional Topics in Trigonometry



6.5 Complex Numbers in Polar Form; DeMoivre's Theorem

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Objectives:

- Plot complex number in the complex plane.
- Find the absolute value of a complex number.
- Write complex numbers in polar form.
- Convert a complex number from polar to rectangular form.
- Find products of complex numbers in polar form.
- Find powers of complex numbers in polar form.
- Find roots of complex numbers in polar form.

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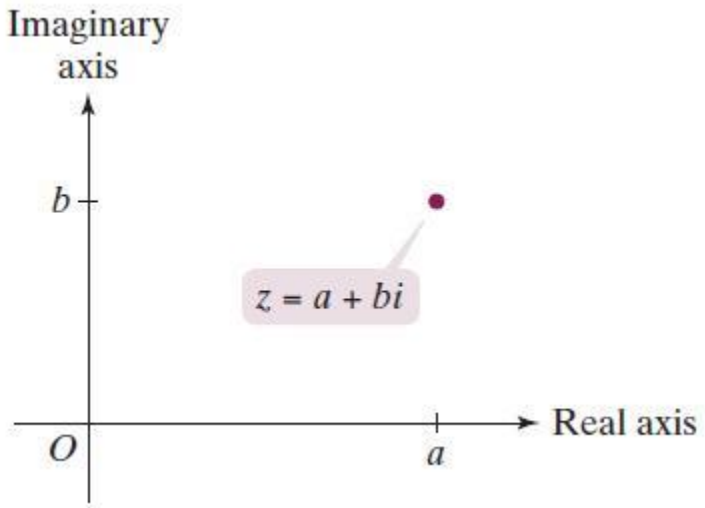
The Complex Plane

A complex number $z = a + bi$ is represented as a point (a, b) in a coordinate plane. The horizontal axis of the coordinate plane is called the **real axis**. The vertical axis is called the **imaginary axis**. The coordinate system is called the **complex plane**.

When we represent a complex number as a point in the complex plane, we say that we are **plotting the complex number**.



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Example: Plotting Complex Numbers

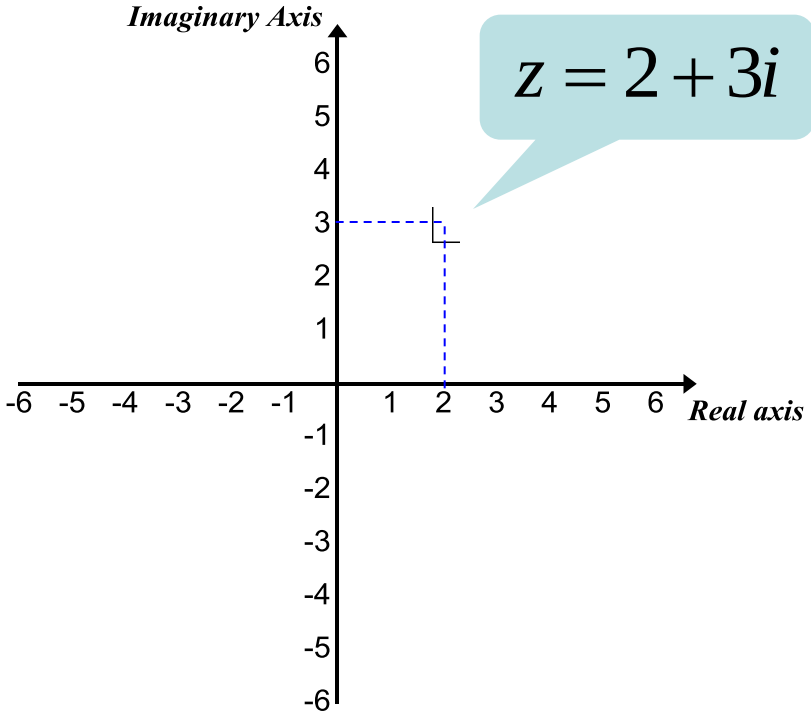
Plot the complex number in the complex plane: $z = 2 + 3i$

$$z = a + bi$$

$$z = 2 + 3i$$

$$a = 2, b = 3$$

We plot the point $(a, b) = (2, 3)$.



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Example: Plotting Complex Numbers

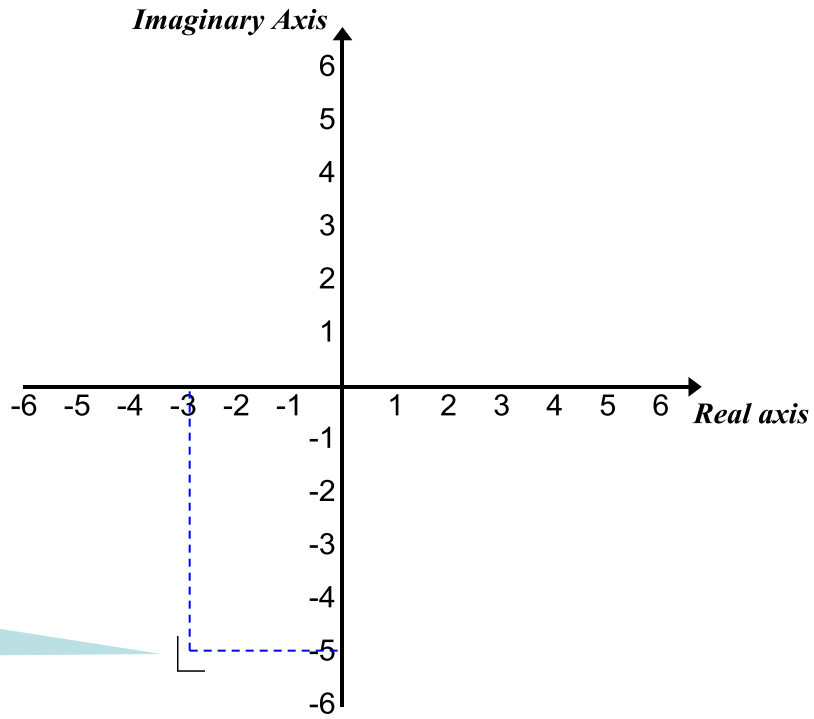
Plot the complex number in the complex plane: $z = -3 - 5i$

$$z = a + bi$$

$$z = -3 - 5i$$

$$a = -3, b = -5$$

We plot the point $(a, b) = (-3, -5)$.



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Example: Plotting Complex Numbers

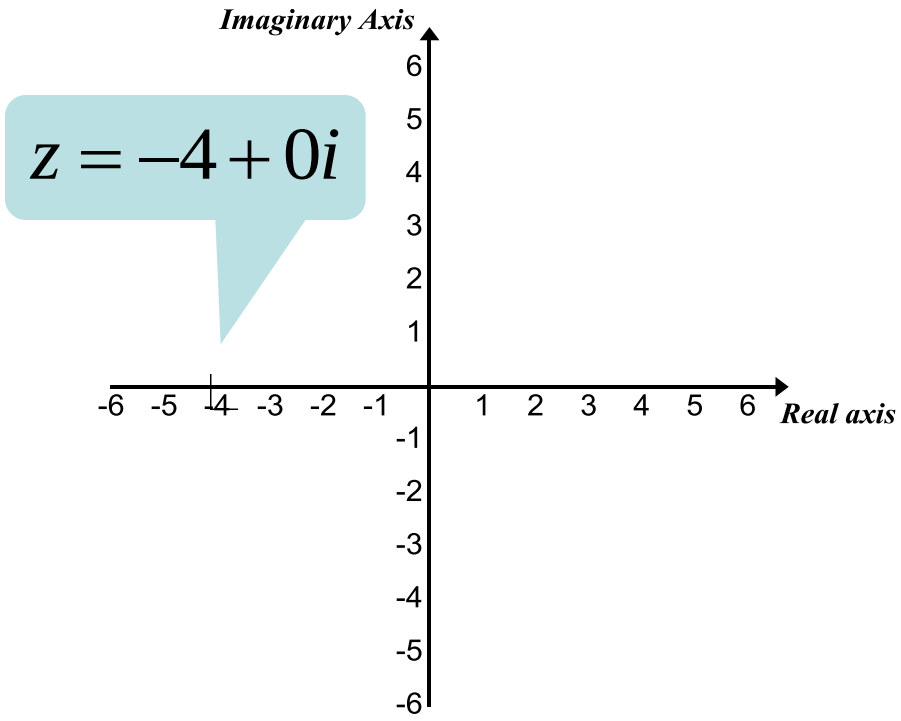
Plot the complex number in the complex plane: $z = -4$

$$z = a + bi$$

$$z = -4 + 0i$$

$$a = -4, b = 0$$

We plot the point $(a, b) = (-4, 0)$.



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Example: Plotting Complex Numbers

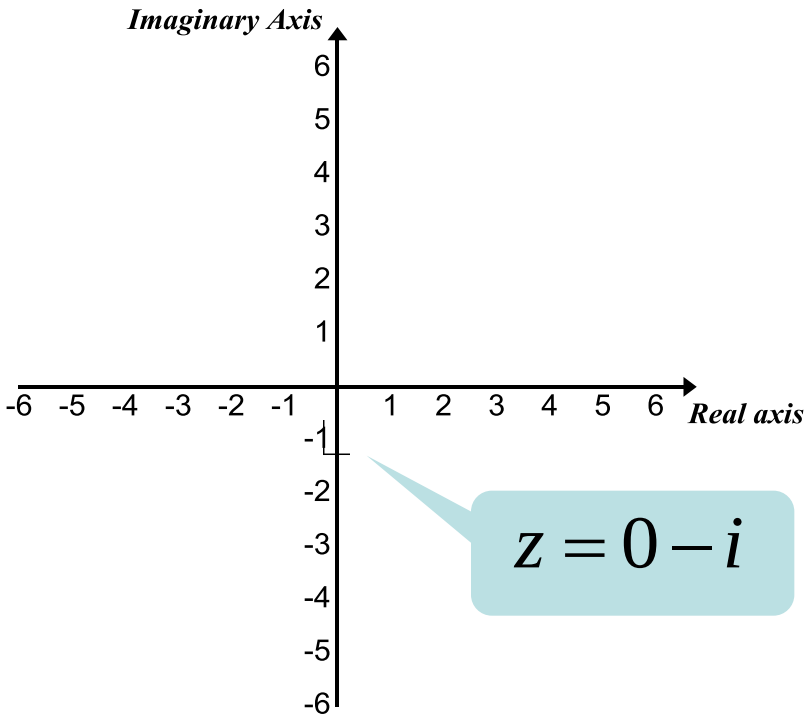
Plot the complex number in the complex plane: $z = -i$

$$z = a + bi$$

$$z = 0 - i$$

$$a = 0, b = -1$$

We plot the point $(a, b) = (0, -1)$.



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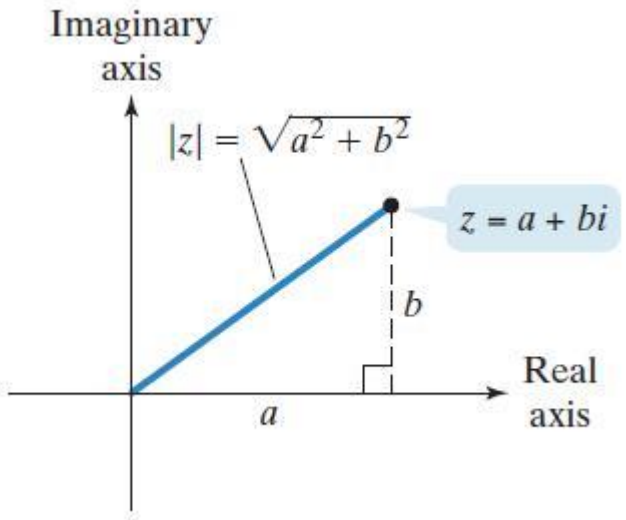
The Absolute Value of a Complex Number

The absolute value of the complex number $z = a + bi$ is the distance from the origin to the point z in the complex plane.

The absolute value of the complex number $a + bi$ is

$$|z| = |a + bi| = \sqrt{a^2 + b^2}.$$



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Example: Finding the Absolute Value of a Complex Number

Determine the absolute value of the following complex number: $z = 5 + 12i$

$$z = 5 + 12i$$

$$|z| = |a + bi| = \sqrt{a^2 + b^2}.$$

$$|z| = |5 + 12i| = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13$$

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Example: Finding the Absolute Value of a Complex Number

Determine the absolute value of the following complex number: $z = 2 - 3i$

$$z = 2 - 3i$$

$$|z| = |a + bi| = \sqrt{a^2 + b^2}.$$

$$|z| = |2 - 3i| = \sqrt{2^2 + (-3)^2} = \sqrt{4 + 9} = \sqrt{13}$$

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Polar Form of a Complex Number

A complex number in the form $z = a + bi$ is said to be in **rectangular form**.

The expression $z = r(\cos\theta + i\sin\theta)$ is called the **polar form of a complex number**.

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Polar Form of a Complex Number *(continued)*

The complex number $z = a + bi$ is written in **polar form** as

$$z = r(\cos\theta + i\sin\theta),$$

where $a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2 + b^2}$, and $\tan\theta = \frac{b}{a}$. The value of r is called the **modulus** (plural: moduli) of the complex number z and the angle θ is called the **argument** of the complex number z with $0 \le \theta < 2\pi$.

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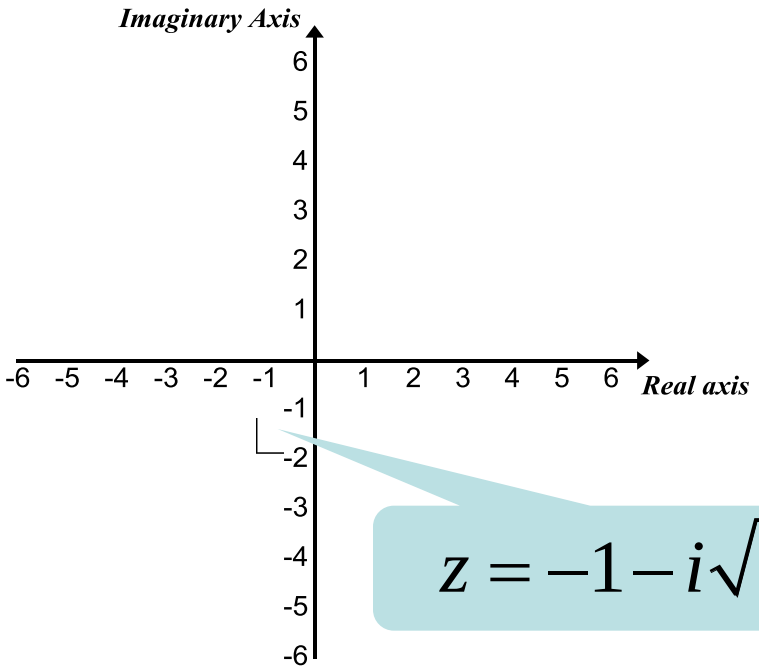
Example: Writing a Complex Number in Polar Form

Plot the complex number in the complex plane, then write the number in polar form: $z = -1 - i\sqrt{3}$

$$z = a + bi$$

$$z = -1 - i\sqrt{3}$$

$$a = -1, b = -\sqrt{3}$$



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Example: Writing a Complex Number in Polar Form

Plot the complex number in the complex plane, then write the number in polar form: $z = -1 - i\sqrt{3}$

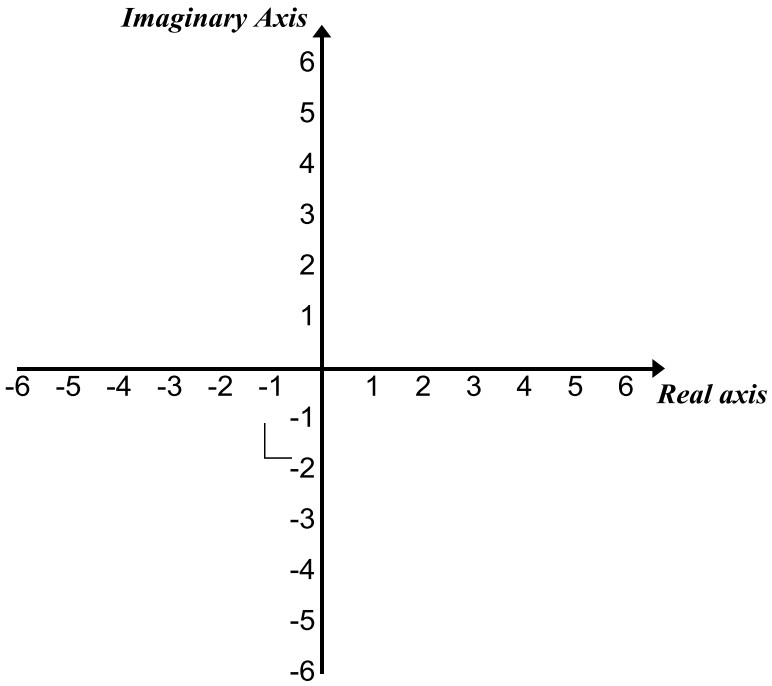
$$r = \sqrt{a^2 + b^2}$$

$$= \sqrt{(-1)^2 + \left(-\sqrt{3}\right)^2}$$

$$= \sqrt{1+3} = \sqrt{4} = 2$$

$$\tan\theta = \frac{b}{a} = \frac{-\sqrt{3}}{-1} = \sqrt{3}$$

$$\theta = \frac{4\pi}{3}$$



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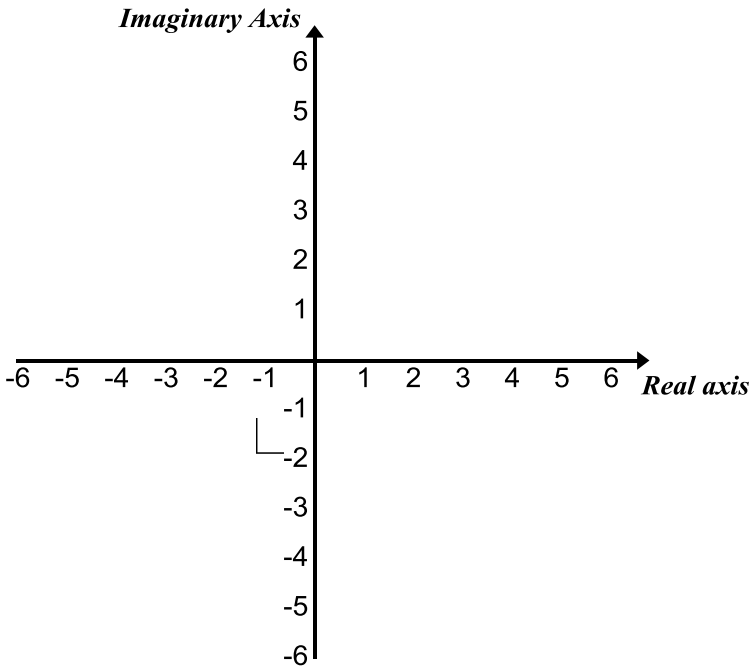
Example: Writing a Complex Number in Polar Form

Plot the complex number in the complex plane, then write the number in polar form: $z = -1 - i\sqrt{3}$



$$r = 2, \theta = \frac{4\pi}{3}$$

$$z = r(\cos\theta + i\sin\theta)$$

The polar form of $z = -1 - i\sqrt{3}$ is $z = 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$.

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The Product of Two Complex Numbers in Polar Form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers in polar form. Their product, z_1z_2, is

$$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

To multiply two complex numbers, multiply moduli and add arguments.

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Example: Finding Products of Complex Numbers in Polar Form

Find the product of the complex numbers. Leave the answer in polar form.

$z_1 = 6(\cos 40° + i \sin 40°)$ $\qquad$ $z_2 = 5(\cos 20° + i \sin 20°)$

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$$

$$z_1 z_2 = (6 \cdot 5)[\cos(40° + 20°) + i \sin(40° + 20°)]$$

$$= 30(\cos 60° + i \sin 60°)$$

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The Quotient of Two Complex Numbers in Polar Form

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ be two complex numbers in polar form. Their quotient, $\frac{z_1}{z_2}$, is

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)].$$

To divide two complex numbers, divide moduli and subtract arguments.

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Example: Finding Quotients of Complex Numbers in Polar Form

Find the quotient of the following complex numbers. Leave the answer in polar form.

$$z_1 = 50\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) \qquad z_2 = 5\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$

$$\frac{z_1}{z_2} = \frac{50}{5}\left[\cos\left(\frac{4\pi}{3} - \frac{\pi}{3}\right) + i\sin\left(\frac{4\pi}{3} - \frac{\pi}{3}\right)\right] = 10(\cos\pi + i\sin\pi)$$

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Powers of Complex Numbers in Polar Form

Let $z = r(\cos\theta + i\sin\theta)$ be a complex number in polar form. If n is a positive integer, then z to the nth power, z^n, is

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta).$$

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Example: Finding the Power of a Complex Number

Find $[2(\cos 30° + i \sin 30°)]^5$. Write the answer in rectangular form, $a + bi$.

$$z^n = r^n(\cos n\theta + i \sin n\theta)$$

$$[2(\cos 30° + i \sin 30°)]^5 = 2^5\left[\cos(5 \cdot 30°) + i \sin(5 \cdot 30°)\right]$$

$$= 32(\cos 150° + i \sin 150°)$$

$$= 32\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$= -16\sqrt{3} + 16i$$

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DeMoivre's Theorem for Finding Complex Roots

Let $w = r(\cos\theta + i\sin\theta)$ be a complex number in polar form. If $w \neq 0$, w has n distinct complex nth roots given by the formula

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right] \quad \text{(radians)}$$

$$\text{or} \quad z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360°k}{n}\right) + i\sin\left(\frac{\theta + 360°k}{n}\right)\right] \text{(degrees)},$$

where $k = 0, 1, 2, \ldots, n - 1$.

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Example: Finding the Roots of a Complex Number

Find all the complex fourth roots of $16(\cos 60° + i\sin 60°)$. Write roots in polar form, with θ in degrees.

There are exactly four fourth roots of the given complex number. The four fourth roots are found by substituting 0, 1, 2, and 3 for k in the expression

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360°k}{n}\right) + i\sin\left(\frac{\theta + 360°k}{n}\right)\right]$$

$$z_0 = \sqrt[4]{16}\left[\cos\left(\frac{60^\circ + 360°\cdot 0}{4}\right) + i\sin\left(\frac{60^\circ + 360°\cdot 0}{4}\right)\right]$$

$$= 2(\cos 15° + i\sin 15°)$$

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Example: Finding the Roots of a Complex Number

Find all the complex fourth roots of $16(\cos 60° + i\sin 60°)$.

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360°k}{n}\right) + i\sin\left(\frac{\theta + 360°k}{n}\right)\right]$$

$$z_1 = \sqrt[4]{16}\left[\cos\left(\frac{60° + 360°\cdot 1}{4}\right) + i\sin\left(\frac{60° + 360°\cdot 1}{4}\right)\right]$$

$$= 2\left(\cos\frac{420°}{4} + i\sin\frac{420°}{4}\right) = 2(\cos 105° + i\sin 105°)$$

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Example: Finding the Roots of a Complex Number

Find all the complex fourth roots of $16(\cos 60° + i\sin 60°)$.

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360°k}{n}\right) + i\sin\left(\frac{\theta + 360°k}{n}\right)\right]$$

$$z_2 = \sqrt[4]{16}\left[\cos\left(\frac{60° + 360°\cdot 2}{4}\right) + i\sin\left(\frac{60° + 360°\cdot 2}{4}\right)\right]$$

$$= 2\left(\cos\frac{780°}{4} + i\sin\frac{780°}{4}\right) = 2(\cos 195° + i\sin 195°)$$

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Example: Finding the Roots of a Complex Number

Find all the complex fourth roots of $16(\cos 60° + i\sin 60°)$.

$$z_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360°k}{n}\right) + i\sin\left(\frac{\theta + 360°k}{n}\right)\right]$$

$$z_3 = \sqrt[4]{16}\left[\cos\left(\frac{60° + 360°\cdot 3}{4}\right) + i\sin\left(\frac{60° + 360°\cdot 3}{4}\right)\right]$$

$$= 2\left(\cos\frac{1140°}{4} + i\sin\frac{1140°}{4}\right) = 2(\cos 285° + i\sin 285°)$$

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Example: Finding the Roots of a Complex Number

Find all the complex fourth roots of $16(\cos 60° + i\sin 60°)$. Write roots in polar form, with θ in degrees.

The four complex fourth roots are:

$$z_0 = 2(\cos 15° + i\sin 15°)$$

$$z_1 = 2(\cos 105° + i\sin 105°)$$

$$z_2 = 2(\cos 195° + i\sin 195°)$$

$$z_3 = 2(\cos 285° + i\sin 285°)$$

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